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A method of evaluating integrals of $r^m\, e^{-\alpha r} j_n(Qr)$

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Abstract. A method for evaluating these integrals in closed form is presented.

In the course of doing some calculations on shallow impurities in semiconductors, it was necessary to evaluate numerous integrals of the form $\int_0^\infty r^m\, e^{-\alpha r} j_n(Qr)\, dr$, where m and n are integers, and j_n is a spherical Bessel function of the first kind. A related integral has been evaluated in general form (Gradshteyn and Ryzhik 1965):

$$\int_0^\infty r^{\mu-1} e^{-\alpha r} J_\nu(Qr)\, dr = \frac{(Q/2\alpha)^\nu \Gamma(\nu+\mu)}{\alpha^\mu \Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{\mu+\nu+1}{2}; \nu+1; -\frac{Q^2}{\alpha^2}\right), \tag{1}$$

where $F(\alpha, \beta; \gamma; z)$ is Gauss' hypergeometric function. Since $j_n(z) = (\pi/2z)^{1/2} J_{n+\frac{1}{2}}(z)$, we have

$$\int_0^\infty r^{-n} e^{-\alpha r} j_n(Qr)\, dr = \left(\frac{\pi}{2}\right)^{1/2} \frac{(Q/2)^{n+\frac{1}{2}}}{\alpha \Gamma(n+\frac{3}{2})} F(\tfrac{1}{2}, 1; n+\tfrac{3}{2}; -Q^2/\alpha^2). \tag{2}$$

$F(\alpha, \beta; \gamma; z)$ can be expressed in closed form if m and n are integers. From Abramowitz and Stegun (1965), we obtain

$$\begin{aligned} F(\tfrac{1}{2}, 1; \tfrac{1}{2}; -z^2) &= \cos^2(\tan^{-1} z) \\ F(\tfrac{1}{2}, 1; \tfrac{3}{2}; -z^2) &= (1/z) \tan^{-1} z. \end{aligned} \tag{3}$$

Using the identity (Abramowitz and Stegun 1965)

$$\begin{aligned} F(a, b; c+1; z) = {} & \frac{c(c-1)}{(c-a)(c-b)} \frac{1-z}{z} F(a, b; c-1; z) \\ & - \frac{c[c-1-(2c-a-b-1)z]}{(c-a)(c-b)z} F(a, b; c; z) \end{aligned} \tag{4}$$

one can then generate the functions $F(\frac{1}{2}, 1; n+\frac{3}{2}; z)$ for higher values of n. Let us introduce the notation

$$I_{m,n}(\alpha, Q) = \int_0^\infty e^{-\alpha r} r^m j_n(Qr)\, dr. \tag{5}$$

Having evaluated $I_{-n,n}$ it is straightforward to obtain this integral for other values of m by parametric differentiation:

$$I_{m+1,n}(\alpha, Q) = -\frac{\partial}{\partial\alpha} I_{m,n}(\alpha, Q). \tag{6}$$

References

Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series, and Products* (New York: Academic Press)